

ON A GENERALIZATION OF ZASLAVSKY'S THEOREM FOR HYPERPLANE ARRANGEMENTS

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ABSTRACT. We define arrangements of codimension-1 submanifolds in a smooth, real manifold which generalize arrangements of hyperplanes. When these submanifolds are removed the manifold breaks up into regions, each of which is homeomorphic to an open disc. The aim of this paper is to derive formulas that count the number of regions formed by an arrangement of submanifolds. We achieve this aim by generalizing Zaslavsky's theorem to this setting. We show that this number is determined by the combinatorics of the intersections of these submanifolds.

Consider the problem of counting the number of pieces into which a topological space is divided when a finitely many of its subspaces are removed. We refer to such a problem as the topological dissection problem. This problem has a long history in combinatorial geometry. In 1826, Steiner considered the problem of counting the pieces of a plane cut by a finite collection of lines, circle etc. In 1901, Schläfli obtained a formula for counting the number of regions in a Euclidean space when it is cut by hyperplanes in general position. Subsequently many mathematicians studied various aspects and generalizations of this problem. We refer to [11, Chapter 18] and [12] for more history related to this problem.

An arrangement of real hyperplanes is a finite collection of hyperplanes in a finite dimensional real vector space. The complement of the union of these hyperplanes is disconnected, in fact, an arrangement stratifies the ambient space into open polyhedral cones or faces. The top dimensional faces are called as chambers. Zaslavsky [28, Theorem A] discovered a counting formula for the number of chambers which depends on the intersection data of the arrangement. To be precise, he shows that the number of chambers is equal to the the characteristic polynomial of the arrangement evaluated at -1 . The characteristic polynomial is a Tutte-Grothendieck type invariant of the associated intersection lattice [28, Section 4A]. It is also shown that the f -polynomial (the generating polynomial of the face-counting numbers) is a special case of the Möbius polynomial (which is related to the Tutte polynomial) of the intersection lattice.

A generalization of hyperplane arrangements, called as the arrangements of submanifolds was introduced in [7]. Such an arrangement is a finite collection of locally flat, codimension 1 submanifolds of a given manifold. Similar to the case of hyperplane arrangements the complement of this arrangement is disconnected. The aim of this paper is to generalize Zaslavsky's result to the submanifold arrangements. We prove that the number of connected components of the complement of a submanifold arrangement is determined by the intersections of these submanifolds. Our result is motivated by the techniques used in [10], where the authors generalize Zaslavsky's formula for toric arrangements. During a discussion with Thomas Zaslavsky

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about this generalization, he directed us to his paper [29] in which he generalizes his own results to arbitrary topological spaces. We would like to point out that though we have proved similar results, the techniques used are different.

We start our paper by a quick review of some combinatorial notions, hyperplane arrangements and Zaslavsky's theorem. We introduce the new object of study, arrangements of submanifolds, in Section 1.2. In Section 1.3 we revise the theory of valuations on a poset and the Euler characteristic. In Section 2 we first introduce a generalization of the characteristic polynomial. We then establish a formula that combines the geometry and combinatorics of the intersections and counts the number of chambers. We also compare Zaslavsky's proof in [29] with ours. Finally in Section 3 we look at some particular cases of manifolds and comment about the f -vectors arising from submanifold arrangements.

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1. PRELIMINARIES

First we list the combinatorial notions we need and fix the notation ([27, Chapter 3] is the main reference).

Definition 1.1. A *partially ordered set* (or *poset*) $\mathcal{P} = (\mathcal{P}, \leq)$ is a set together with a relation \leq that satisfies the following three axioms:

- (1) *idempotency*: for any $x \in \mathcal{P}$, we have $x \leq x$;
- (2) *antisymmetry*: for any $x, y \in \mathcal{P}$, if $x \leq y$ and $y \leq x$ then $x = y$;
- (3) *transitivity*: for any $x, y, z \in \mathcal{P}$, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Unless stated otherwise \mathcal{P} is a finite set. The subposet $\mathcal{P}_{\leq x} := \{y \in \mathcal{P} \mid y \leq x\}$ is called the **principal ideal** generated by x . (The notions $\mathcal{P}_{\geq x}, \mathcal{P}_{< x}, \mathcal{P}_{> x}$ are defined analogously.) For $x \leq y$ define the *open interval* $(x, y) := \mathcal{P}_{> x} \cap \mathcal{P}_{< y}$ and the *closed interval* $[x, y] := \mathcal{P}_{\geq x} \cap \mathcal{P}_{\leq y}$. A totally ordered subset $x_0 < x_1 < \dots < x_k$ is called a *chain of length k* . The length of $\mathcal{P}_{\leq x}$ is called *height of x in \mathcal{P}* . Since \mathcal{P} is finite every element in \mathcal{P} has finite height. Such posets are also called as *ranked posets*. The dual \mathcal{P}^* of a poset \mathcal{P} is the poset obtained by reversing the order. A *poset map* is an order preserving (or reversing) function between two posets.

The notion that encodes combinatorial as well as the topological information about a poset \mathcal{P} is the Möbius function $\mu: \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{Z}$. It is defined recursively as follows:

$$\begin{aligned} \mu(x, x) &= 1, \text{ for all } x \in \mathcal{P} \\ \mu(x, y) &= - \sum_{x \leq z < y} \mu(x, z), \text{ for all } x < y \in \mathcal{P} \end{aligned}$$

A poset is called *bounded* if it has a unique minimum element $\hat{0}$ and a unique maximum element $\hat{1}$. For a bounded poset \mathcal{P} , $\mu(\mathcal{P}) := \mu(\hat{0}, \hat{1})$. In case a poset is not bounded then one can define the *augmented poset* as $\hat{\mathcal{P}} := \mathcal{P} \cup \{\hat{0}, \hat{1}\}$, note that if \mathcal{P} is bounded then $(\mathcal{P}, \leq) \cong (\hat{\mathcal{P}}, \leq)$. The Möbius function of a poset is used to obtain inversion formulas (which in some sense generalize the principal of inclusion-exclusion).

Lemma 1.2. *Let P be a poset and let $f, g: P \rightarrow \mathbb{C}$. Then*

$$g(y) = \sum_{x \leq y} f(x)$$

if and only if

$$f(y) = \sum_{x \leq y} \mu(x, y) g(x).$$

Let \mathcal{P} be a finite poset with $\hat{0}$. The **characteristic polynomial** of \mathcal{P} is defined as the finite sum

$$\sum_{x \in \mathcal{P}} \mu(\hat{0}, x) \cdot t^{r(x)}.$$

The absolute value of the coefficient of t^k in the characteristic polynomial is called the *k-th Whitney number (of the second kind)*. Finally a **lattice** is a poset in which any two elements have a unique supremum (called their join and denoted by $x \wedge y$) and an infimum (called their meet and denoted by $x \vee y$).

1.1. Basics of hyperplane arrangements. Hyperplane arrangements arise naturally in geometric, algebraic and combinatorial instances. They occur in various settings such as finite dimensional projective or affine (vector) spaces defined over field of any characteristic. Here we formally define hyperplane arrangements and the combinatorial data associated with it in a setting that is most relevant to our work.

Definition 1.3. A real *arrangement of hyperplanes* is a collection $\mathcal{A} = \{H_1, \dots, H_k\}$ of finitely many hyperplanes in \mathbb{R}^l , $l \geq 1$. l is called as the *rank* of the arrangement.

We call \mathcal{A} a *central* arrangement if all the hyperplanes pass through the origin otherwise we call \mathcal{A} an *affine* arrangement. For an affine subspace X of \mathbb{R}^l , the *contraction* of X in \mathcal{A} is given by the sub-arrangement $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$. The hyperplanes of \mathcal{A} induce a stratification (cellular decomposition) of \mathbb{R}^l , components of each stratum are called *faces*.

There are two posets associated with \mathcal{A} , namely, the face poset and the intersection lattice which contain important combinatorial information about the arrangement.

Definition 1.4. The *intersection lattice* $L(\mathcal{A})$ of a central arrangement \mathcal{A} is defined as the set of all intersections of hyperplanes ordered by reverse inclusion.

$$L(\mathcal{A}) := \{X := \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}, X \neq \emptyset\}, \quad X \geq Y \Leftrightarrow X \subseteq Y$$

The rank of each element is the codimension of the corresponding intersection. The meet of two elements is defined as the smallest subspace in $L(\mathcal{A})$ which contains their union. Whereas the meet of two elements is their intersection. For affine arrangements, set of all intersections do not form a lattice.

Definition 1.5. Let \mathcal{A} be an arrangement with its intersection lattice $L(\mathcal{A})$ and let μ be the Möbius function of the lattice. Define *characteristic polynomial* of \mathcal{A} as

$$\chi(\mathcal{A}, t) := \sum_{X \in L} \mu(X) \cdot t^{\dim(X)}$$

Definition 1.6. The *face poset* $\mathcal{F}(\mathcal{A})$ of \mathcal{A} is the set of all faces ordered by inclusion: $F \leq G$ if and only if $F \subseteq \overline{G}$.

Codimension 0 faces are called *chambers*, the set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$. A chamber is bounded if and only if it is a bounded subset of \mathbb{R}^l . Two chambers C and D are adjacent if they have a common face. As the complement of the hyperplanes in \mathbb{R}^l is disconnected, a natural question is to ask if the number of chambers depend on the intersection data. Zaslavsky in his fundamental treatise [28] studied the relationships between the intersection lattice of an arrangement and the set of chambers. He developed the enumeration theory for hyperplane arrangements by exploiting the combinatorial structure of the intersection lattice. His main result is as follows:

Theorem 1.7 (Theorem A [28]). *Let \mathcal{A} be a hyperplane arrangement in \mathbb{R}^l with $L(\mathcal{A})$ as its intersection lattice and $\chi(\mathcal{A}, t)$ be the associated characteristic polynomial. Then the number of its chambers is*

$$|\mathcal{C}(\mathcal{A})| = \sum_{X \in L(\mathcal{A})} |\mu(\mathbb{R}^l, X)| = |(-1)^l \chi(\mathcal{A}, -1)|.$$

It is important to note here that the face poset of a central arrangement corresponds to a realizable oriented matroid. Under this correspondence the intersection lattice corresponds to the underlying matroid (i.e. the lattice of flats). In the general setting of oriented matroids the above formula counts the number topes (see [2, Section 4.6] for details). The definition of the characteristic polynomial (or more generally that of the Tutte polynomial) for central arrangements coincides with the usual definition of these polynomials for matroids.

1.2. Submanifold arrangements. In this section we propose a generalization of arrangements of hyperplanes. In order to achieve this generalization we isolate the following characteristics of a hyperplane arrangement:

- (1) there are finitely many codimension 1 subspaces (with nonempty intersections) separating the ambient space,
- (2) there is a stratification of the ambient topological space into contractible subsets,
- (3) (the geometric realization of) the face poset (of this stratification) has the homotopy type of the ambient space.

Any reasonable generalization of hyperplane arrangements should possess these properties. Since smooth manifolds are locally Euclidean they are obvious candidates for the ambient space. In this setting we can study arrangements of codimension 1 submanifolds that satisfy certain nice conditions, for example, locally, we would like our submanifolds to behave like hyperplanes.

Our focus is on the codimension 1 smooth submanifolds that are *embedded as a closed subset* of a finite dimensional smooth manifold. These types of submanifolds behave much like hyperplanes. The following are some of their well known properties.

Lemma 1.8. *If X is a connected l -manifold and N is a connected $(l-1)$ -manifold embedded in X as a closed subset, then $X \setminus N$ has either 1 or 2 components. If in addition $H_1(X, \mathbb{Z}_2) = 0$ then $X \setminus N$ has two components.*

Proof. The lemma follows from the following exact sequence of pairs in mod 2 homology:

$$H_1(X, \mathbb{Z}_2) \rightarrow H_1(X, X \setminus N, \mathbb{Z}_2) \rightarrow \tilde{H}_0(X \setminus N, \mathbb{Z}_2) \rightarrow \tilde{H}_0(M, \mathbb{Z}_2) = 0. \quad \square$$

Definition 1.9. A connected codimension 1 submanifold N in X is said to be *two sided* if N has a neighborhood U_N such that $U_N \setminus N$ has two connected components; otherwise N is said to be *one sided*. In general a disconnected codimension 1 submanifold is two sided if each of its connected component is two sided. Moreover a submanifold *separates* X if its complement has 2 components.

Note that being two sided is in some sense a local condition. For example, a point in S^1 does not separate S^1 , however, it is two sided. The following corollary follows from the definitions.

Corollary 1.10. *Every codimension 1 submanifold N is locally two sided in X ; that is, each $x \in N$ has arbitrarily small connected neighborhoods U_x such that $U_x \setminus (U_x \cap N)$ has two components.*

Corollary 1.11. *If X is a l -manifold and N is a $(l - 1)$ manifold embedded in X as a closed subset, where $H_1(N, \mathbb{Z}_2) \cong 0$ then N is two sided.*

From now on we assume that a submanifold is always embedded as a closed subset. An n -manifold N contained in the interior of an l -manifold X is *locally flat* at $x \in N^\circ$, if there exists a neighborhood U_x of x in X such that $(U_x, U_x \cap N) \cong (\mathbb{R}^l, \mathbb{R}^n)$. An embedding $f: N \rightarrow X$ such that $f(N^\circ) \subset X^\circ$ is said to be *locally flat at a point* $x \in N$ if $f(N)$ is locally flat at $f(x)$. Embeddings and submanifolds are *locally flat* if they are locally flat at every point.

It is necessary to consider the locally flat class of submanifolds otherwise one could run into pathological situations. For example, the Alexander horned sphere, it is an (non flat) embedding of S^2 inside S^3 such that the connected components of its complement are not even simply connected (see [24, Page 65]). Rushing's book [24] is an excellent reference for more examples of so called *wild* (non flat) embeddings and other important results in the field of topological embeddings. Finally we add a technical condition so that these locally flat submanifolds intersect like hyperplanes.

Definition 1.12. Let X be a manifold of dimension l and let $\{N_1, \dots, N_k\}, (k \geq 2)$ be codimension 1, locally flat submanifolds of X . We say that these submanifolds *intersect like hyperplanes* if and only if for every nonempty $Y \subseteq X$ which can be written as $Y = \cap_{i=1}^j N_i, (j \leq l)$ and for every $x \in Y$ there exists an open neighborhood V_x of x and a coordinate chart $\phi_x: V_x \rightarrow \mathbb{R}^l$ such that for each N_i containing x , the image $\phi_x(N_i \cap V_x)$ is a hyperplane passing through the origin.

The desired generalization of hyperplane arrangements is the following:

Definition 1.13. Let X be a connected, smooth, real manifold of dimension l . An **arrangement of submanifolds** is a finite collection $\mathcal{A} = \{N_1, \dots, N_k\}$ of codimension 1 locally flat smooth submanifolds in X such that

- (1) The N_i 's intersect like hyperplanes.
- (2) $X \setminus N_i$ has exactly two connected components for every i .
- (3) The intersections of N_i 's define a regular CW decomposition of X .

The submanifolds in an arrangement are allowed to have more than one connected component.

Note that in the above definition we can use arbitrary topological manifolds instead of smooth manifolds. However we restrict our attention to smooth manifolds. Let us look at how we can associate combinatorial data to such an arrangement and a few examples. Just like in

the case of hyperplane arrangements, the intersection sets determined by these submanifolds have a combinatorial structure.

Definition 1.14. The *intersection poset* denoted by $L(\mathcal{A})$ is the set of connected components of all possible intersections of N_i 's ordered by reverse inclusion, by convention $X \in L(\mathcal{A})$ as the smallest element. The rank of each element in $L(\mathcal{A})$ is defined to be the codimension of the corresponding intersection.

However note that (similar to the case of affine hyperplane arrangements) in general this poset need not be a geometric lattice. Also, the use of connected components of intersections is not a new idea, see for example [29] and more recently [9, 18] (in case of toric arrangements). Now we move on to another poset associated with such an arrangement.

Definition 1.15. The intersections of these N_i 's in \mathcal{A} define a stratification of X . The connected components in each stratum are called *faces*. Top dimensional faces are called chambers and the set of all chambers is denoted by $\mathcal{C}(\mathcal{A})$. The collection of all the faces $\mathcal{F}(\mathcal{A}) = \cup \mathcal{F}^i(\mathcal{A})$ is the *face poset* with the ordering $F \leq G \Leftrightarrow F \subseteq \overline{G}$. It is a graded poset and the rank of each face is its dimension.

The obvious examples of these submanifold arrangements are the hyperplane arrangements in some Euclidean space. Here are some different types of arrangements.

Example 1.16. Let X be the circle S^1 , a smooth one dimensional manifold, the codimension 1 submanifolds are points in S^1 . Consider the arrangement $\mathcal{A} = \{p, q\}$ of 2 points. For both these points there is an open neighborhood which is homeomorphic to an arrangement of a point in \mathbb{R} . Figure 1 shows this arrangement and the Hasse diagrams of the face poset and the intersection poset.

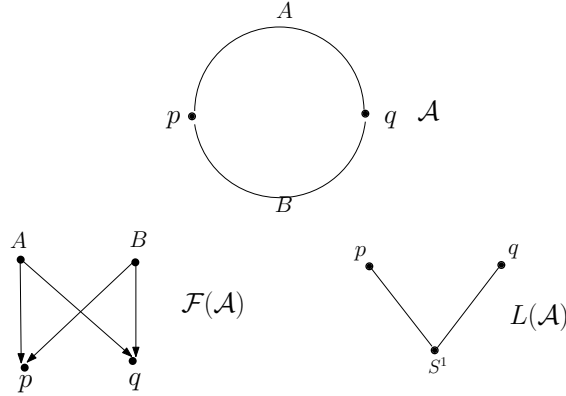


FIGURE 1. Arrangement of 2 points in a circle.

Example 1.17. As a 2-dimensional example consider an arrangement of 2 great circles N_1, N_2 in S^2 . Figure 2 shows this arrangement and the related posets. The face poset has two 0-cells, four 1-cells and four 2-cells. Also note that the order complex of the face poset has the homotopy type of S^2 .

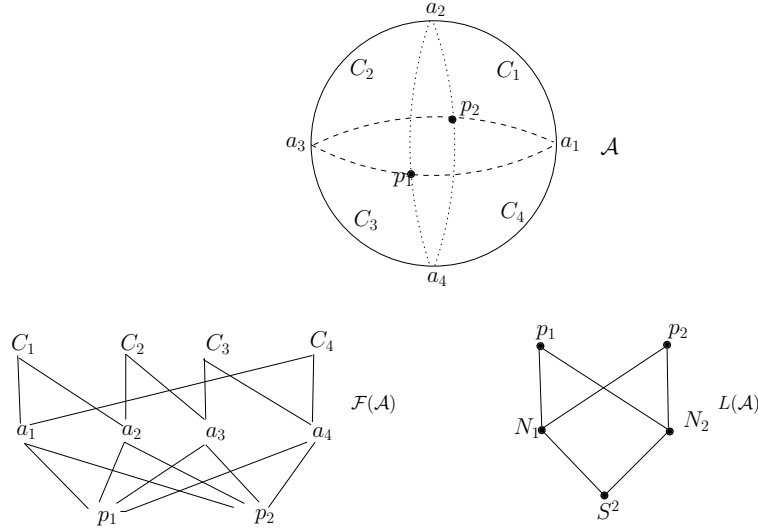


FIGURE 2. Arrangement of 2 circles in a sphere.

Recently there has been an increase in interest to study arrangements in a torus (see for example [6, 9, 16, 18–20]). We claim that these toric arrangements are examples of submanifold arrangements. We refer the reader to [7, Chapter 4] for an explicit treatment of toric arrangements, arrangements in spheres and projective spaces.

1.3. Measure theory for posets. In his attempt to classify the convex polyhedra in 3-space, Leonhard Euler discovered that the number of vertices of a polyhedron minus the number of its edges plus the number of its faces is an invariant. He published a proof of this result in 1758 and also conjectured that the result is true for higher dimensional polytopes. The invariant popularly known as the *Euler characteristic* not only appears in many branches of mathematics but plays an important role in those areas. A century later Schläfli proved the *Euler relation* for polytopes of higher dimensions and Poincaré, using homology theory, extended the result to manifolds. For more on the history of the Euler characteristic see [11, Chapter 8] and [8].

Our focus here is the combinatorial nature of this invariant. In 1955 Hadwiger [13] characterized the Euler characteristic as the unique translation-invariant, finitely additive set function defined on finite unions of compact convex subsets of \mathbb{R}^n . Inspired by Hadwiger's work, Victor Klee gave a relatively elementary proof of the Euler formula for polytopes in [15]. Motivated by these results Rota in [23] established a combinatorial connection between the Euler characteristic and underlying order-theoretic structure. His work revealed that the Euler characteristic can be thought of as a fundamental dimension-less invariant, associated with any mathematical structure, that can be defined in much more general context. For example, in case of a finite set the Euler characteristic is its cardinality. Generalizing this basic idea, Schanuel [25] showed that the Euler characteristic of certain polyhedra is determined by a simple universal property. A further generalization is achieved by Leinster in [17] by defining the Euler characteristic for finite categories.

The unique universal property of the Euler characteristic will be used to generalize Zaslavsky's result. We devote this section to the introduction of the theory of valuations on

lattices and also explain how the Euler characteristic is defined combinatorially. The main references for this introductory material are [23] and [14, Chapter 2].

Let \mathcal{D} be a family of subsets of a set S such that \mathcal{D} is closed under finite unions and finite intersections. Such a family is a distributive lattice in which the partial ordering is given by the inclusion of subsets, empty set is the least element $\hat{0}$, S is the greatest element while the meet and join are defined by intersection and union of subsets, respectively. All of the following theory holds true for arbitrary distributive lattices but we state it in the context that best suits our purpose. Let R be a commutative ring with 1.

Definition 1.18. An R -valuation on \mathcal{D} is a function $\nu: \mathcal{D} \rightarrow R$, satisfying

$$(1.1) \quad \nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B)$$

$$(1.2) \quad \nu(\emptyset) = 0$$

By iterating the identity (1.1) we get the *inclusion-exclusion principle* for ν , namely

$$(1.3) \quad \nu(A_1 \cup \cdots \cup A_n) = \sum_i \nu(A_i) - \sum_{i < j} \nu(A_i \cap A_j) + \sum_{i < j < k} \nu(A_i \cap A_j \cap A_k) + \cdots.$$

The above definition is clearly similar to that of a measure on a Boolean algebra. But the theory of valuations is in some sense richer, for example, the Euler characteristic (which we will show is a valuation) has no counterpart in measure theory. In functional analysis a measure is regarded either as a linear functional or as an abstract integral on a function space. Drawing parallels with this aspect, Rota in [23] defined a ring for distributive lattices called the *valuation ring* (see Definition 2.9 below), denoted by $V(\mathcal{D}, R)$ and identified R -valuations on \mathcal{D} with R -valued functionals on $V(\mathcal{D}, R)$. Moreover when the distributive lattice is finite any valuation can be uniquely determined by the following theorem. Recall that in a lattice a join-irreducible element cannot be written as a join of two distinct elements.

Theorem 1.19 (Rota [23]). *A valuation on a finite distributive lattice \mathcal{D} is uniquely determined by the values it takes on the set of join-irreducible elements of \mathcal{D} , and these values can be arbitrarily assigned.*

With this theorem we are now in a position to define the Euler characteristic.

Definition 1.20. The *Euler characteristic* of a finite distributive lattice \mathcal{D} is the unique valuation χ such that $\chi(x) = 1$ for all join-irreducible elements x and $\chi(\hat{0}) = 0$.

Before moving on let us look at some examples. Recall that for a finite set S an abstract simplicial complex \mathcal{E} is a family of subsets of S such that if $A \in \mathcal{E}$ and $B \subseteq A$ then $B \in \mathcal{E}$. The containment of subsets induces a partial order on \mathcal{E} , simplicial complex with exactly one maximal element is called as a simplex. The set of all possible (abstract) simplicial complexes forms a distributive lattice.

Example 1.21. Let S be a finite set and \mathcal{D} be the lattice consisting of all abstract simplicial complexes defined on the power set of S . In this case the Euler characteristic defined above coincides with the classical definition. Note that the join-irreducible elements of \mathcal{D} are the simplices hence their Euler characteristic is exactly 1. Moreover, for $A \in \mathcal{D}$ let f_i denotes the number of cardinality i subsets, then

$$\chi(A) = \sum_{i \geq 0} (-1)^i f_i.$$

The above example justifies the name Euler characteristic and the next examples show how it is defined in a different context and for infinite lattices.

Example 1.22. Let \mathcal{D} denote the lattice of positive integers, ordered by divisibility, the join operation defined as the product and the meet operation defined as gcd. Then, one can show that for a positive integer n its Euler characteristic $\chi(n)$ is equal to the number of distinct prime divisors of n .

Example 1.23. Let \mathcal{D} be the lattice generated by n -dimensional polytopes in \mathbb{R}^n . In this lattice the join-irreducible elements are the compact convex polytopes and the unique \mathbb{R} -valuation that takes value 1 on them is the (classical) Euler characteristic. Moreover if we consider the lattice generated by all relative interiors of convex n -polytopes then the definition extends to polytopal complexes and non-closed convex sets. For the proof and other details see [14, Chapter 5] and [26, Chapter 3].

Just like measures, a valuation on a lattice can be used to construct abstract integrals. Since integrals with respect to a valuation will be used to generalize the characteristic polynomial we will briefly sketch some important facts. For simplicity, from now on let D be a finite distributive lattice consisting of subsets of a finite set S .

A D -simple function $f: S \rightarrow R$ is defined by the following finite linear combination

$$(1.4) \quad f(x) := \sum_{i=1}^k r_i I_{A_i}(x) \quad \forall x \in S$$

where $r_i \in R$ and $I_{A_i}: S \rightarrow \{0, 1\}$ are the *indicator (or characteristic) functions* (i.e. $I_{A_i}(x) = 1$ if $x \in A_i$ and 0 otherwise) with $A_i \in \mathcal{D}$ for $1 \leq i \leq k$. The set of all \mathcal{D} -simple functions forms a ring under point wise addition and multiplication. Note that unlike the valuations, these simple functions are defined on S .

A subset L of \mathcal{D} is called a *generating set* if it is closed under finite intersections and if every element of \mathcal{D} can be expressed as a finite union of members of L . Using the inclusion-exclusion formula for the indicator functions it can be shown that every \mathcal{D} -simple function can be rewritten as a linear combination

$$(1.5) \quad f = \sum_{i=1}^m s_i I_{B_i}$$

where each $B_i \in L$. An R -valued function ν on L is called a *valuation* on L provided that ν satisfies the identities (1.1) and (1.2) for all sets $A, B \in L$ such that $A \cup B \in L$. For a \mathcal{D} -simple function f , define the *integral* of f with respect to ν as

$$(1.6) \quad \int f d\nu = \sum_{i=1}^m s_i \nu(B_i)$$

The existence of the extension of ν to \mathcal{D} and also the existence of the integral are equivalent properties of ν . This nontrivial fact is stated as the following -

Theorem 1.24. (Groemer's Integral Theorem) *Let L be a generating set for a lattice \mathcal{D} , and let ν be a valuation on L . Then the following are equivalent :*

- (1) ν extends uniquely to a valuation on \mathcal{D} .

(2) ν satisfies the inclusion-exclusion identities

$$(1.7) \quad \nu(B_1 \cup \dots \cup B_n) = \sum_i \nu(B_i) - \sum_{i < j} \nu(B_i \cap B_j) + \sum_{i < j < k} \nu(B_i \cap B_j \cap B_k) + \dots$$

whenever $B_i \in L \forall i$ and $B_1 \cup \dots \cup B_n \in L$, for all $n \geq 2$.

(3) ν defines an integral on the R -algebra of \mathcal{D} -simple functions.

Proof. See [14, Theorem 2.2.7] □

2. THE CHAMBER COUNTING FORMULA

Let \mathcal{A} be an arrangement of submanifolds of a smooth l -manifold X . The problem at hand is to count the number of connected components of the complement $X \setminus \bigcup_{N \in \mathcal{A}} N$, which will be denoted by $|\mathcal{C}(\mathcal{A})|$. Let \mathcal{D} be the lattice of sets generated by the intersection poset $L(\mathcal{A})$ and the members of $\mathcal{C}(\mathcal{A})$ through finite unions and finite intersections. Recall that to count the number of chambers of a hyperplane arrangement, we use the *characteristic polynomial* of the intersection lattice (Theorem 1.7). We start by generalizing this polynomial.

Define the *Poincaré polynomial with compact support* of a topological space A as

$$Poin_c(A, t) := \sum_{i \geq 0} \text{rank}(H_c^i(A, \mathbb{Z})) t^i$$

where H_c^i is the cohomology with compact supports.

Lemma 2.1. *The function $\nu: \mathcal{D} \rightarrow \mathbb{Z}[t]$ defined by $\nu(A) = Poin_c(A, t)$, $\forall A \in \mathcal{D}$ is a valuation on \mathcal{D} .*

Proof. The first step is to find a generating set for \mathcal{D} . Let G denote the union of all faces of \mathcal{A} and the empty set \emptyset . Clearly the collection G is closed under intersections. Each member of $L(\mathcal{A})$ is a union of finitely many faces and as all the chambers are faces any finite union (or intersection) of members of $L(\mathcal{A})$ and $\mathcal{C}(\mathcal{A})$ can be expressed as a finite union of faces. This observation proves that G is a generating set for \mathcal{D} .

Now we have to show that ν defines a valuation on G . This is clear because $\nu(\emptyset) = 0$ and all the non-empty faces are disjoint, open topological cells. For the same reasons the inclusion-exclusion identities (1.7) are satisfied by ν . Hence as a consequence of *Groemer's integral theorem* (Theorem 1.24), ν extends to whole of \mathcal{D} . □

For each $Y \in L (= L(\mathcal{A}))$, define

$$f(Y) = Y \setminus \bigcup_{\substack{Y < Z \\ Z \in L}} Z$$

Then $\{f(Y) \mid Y \in L\}$ is the set of all faces of the arrangement, in particular it is a disjoint collection. Hence

$$\begin{aligned} X &= \coprod_{\substack{X \leq Y \\ Y \in L}} f(Y) \\ \text{so } I_X &= \sum_{X \leq Y} I_{f(Y)} \\ (2.1) \quad \text{and } I_{f(X)} &= \sum_{X \leq Y} \mu(X, Y) I_Y \quad (\text{by Möbius inversion}) \end{aligned}$$

Note that $f(X)$ is in fact the union of all the chambers, therefore $I_{f(X)}$ is a \mathcal{D} -simple function. As ν extends uniquely to a valuation on \mathcal{D} , it defines an integral on the algebra of \mathcal{D} -simple functions. Integrating $I_{\mathcal{C}(\mathcal{A})}$ with respect to ν , we get

$$\begin{aligned} \int I_{\mathcal{C}(\mathcal{A})} d\nu &= \sum_{Y \in L} \mu(X, Y) \nu(Y) \\ (2.2) \qquad &= \sum_{Y \in L} \mu(X, Y) \text{Poin}_c(Y, t) \end{aligned}$$

Definition 2.2. Let \mathcal{A} be an arrangement of submanifolds in an l -manifold X and L be the associated intersection poset. The *generalized characteristic polynomial* of \mathcal{A} is

$$\chi(\mathcal{A}, t) := \sum_{Y \in L} \mu(X, Y) \text{Poin}_c(Y, t)$$

Note the unfortunate clash of notations.

Definition 2.3. The *combinatorial Euler characteristic* κ of a finite CW complex P is defined as

$$\kappa(P) = \begin{cases} \chi(\hat{P}) - 1 & \text{if } P \text{ is not compact} \\ \chi(P) & \text{if } P \text{ is compact} \end{cases}$$

where $\chi(\hat{P})$ is the Euler characteristic of the one-point compactification of P .

Note that this is not a new notion, this is just a topological description of Definition 1.20. To give an intrinsic topological description of the combinatorial Euler characteristic for arbitrary spaces is not an easy job. The theory of o-minimal structures has to be used in order to define valuations and integrals on arbitrary spaces, which is beyond the scope of this paper. However, the above notion is a topological invariant and it satisfies the Euler relation, that is the number of even dimensional cells minus the number of odd dimensional cells is equal to the κ value. The following lemma is now clear.

Lemma 2.4. *The combinatorial Euler characteristic κ defines an \mathbb{R} -valuation on D .*

Using all of the theory developed so far we can now generalize Zaslavsky's theorem.

Theorem 2.5. *Let \mathcal{A} be an arrangement of submanifolds in an l -manifold X (that is \mathcal{A} subdivides the manifold into chambers homeomorphic to open l -dimensional balls). Then the number of chambers is given by*

$$(-1)^l \sum_{Y \in L} \mu(X, Y) \kappa(Y),$$

where μ is the Möbius function of the intersection poset and κ is the combinatorial Euler characteristic.

Proof. First note that κ and $\nu|_{t=-1}$ agree on every element of G . Consequently, they also agree on every member of D . Hence, we have that

$$(2.3) \qquad \kappa(\mathcal{C}(\mathcal{A})) = \int I_{\mathcal{C}(\mathcal{A})} d\kappa = \int I_{\mathcal{C}(\mathcal{A})} d\nu|_{t=-1}$$

The set $\mathcal{C}(\mathcal{A})$ is a disjoint union of chambers, each of which is homeomorphic to an open ball of dimension l . Consequently, the combinatorial Euler characteristic of a chamber is $(-1)^l$, substituting this in the equation 2.3, we get

$$\begin{aligned}
 |\mathcal{C}(\mathcal{A})| &= (-1)^l \kappa(\mathcal{C}(\mathcal{A})) \\
 &= (-1)^l \int I_{\mathcal{C}(\mathcal{A})} d\nu|_{t=-1} \\
 &= (-1)^l \sum_{Y \in L} \mu(X, Y) \text{Poin}_c(Y, -1) \\
 (2.4) \qquad &= (-1)^l \sum_{Y \in L} \mu(X, Y) \kappa(Y) \quad \square
 \end{aligned}$$

Corollary 2.6. (*Zaslavsky [28]*) *Let \mathcal{A} be a hyperplane arrangement in \mathbb{R}^n . The number of chambers of this arrangement is equal to $(-1)^n \chi(\mathcal{A}, -1)$.*

Proof. Every member of the intersection poset in this case is homeomorphic to an open ball, hence $\text{Poin}_c(Y, t) = t^{\dim Y}$ for every $Y \in L(\mathcal{A})$. The result follows from the observation that $\int I_{\mathcal{C}(\mathcal{A})} d\nu = \chi(\mathcal{A}, t)$. \square

Corollary 2.7. (*Ehrenborg et al. [9]*) *For a toric hyperplane arrangement \mathcal{A} in a torus T^n that subdivides the torus into open n -dimensional balls, the number of chambers is given by*

$$(-1)^n \sum_{\dim Y=0} \mu(T^n, Y)$$

Proof. Note that toric hyperplanes are homeomorphic to T^{n-1} and they intersect into lower dimensional subtori. Hence the elements of the intersection poset L are tori of some finite dimension, except for the coatoms that are points. If $Y \in L$ is k -dimensional torus then $\text{Poin}_c(Y, t) = (1+t)^k$.

$$\begin{aligned}
 \int I_{\mathcal{C}(\mathcal{A})} d\nu &= \sum_{1 \leq \dim Y \leq n} \mu(T^n, Y) (1+t)^{\dim Y} + \sum_{\dim Y=0} \mu(T^n, Y) \\
 \Rightarrow |\mathcal{C}(\mathcal{A})| &= (-1)^n \sum_{\dim Y=0} \mu(T^n, Y)
 \end{aligned}$$

\square

Though the Theorem 2.5 is stated for submanifold arrangements it is valid in a more general context. The only thing we have used in the proof is that the combinatorial Euler characteristic is a valuation. Now consider a more general situation where X is a topological space and \mathcal{A} is a finite collection of subspaces that are removed from X . Let L denote the poset consisting of X and connected components of the all possible finite intersections of members of \mathcal{A} ordered by reverse inclusion. The topological dissection problem asks whether it is possible to express the number of connected components of the complement in terms of L (dissection of X intuitively means that X is expressed as a union of pairwise disjoint subspaces). Let $\{C_1, \dots, C_m\}$ denote the connected components of the complement of union of members of \mathcal{A} , in this context Theorem 2.5 takes the following form

Theorem 2.8. *If the combinatorial Euler characteristic κ is a valuation on the lattice of sets generated by $L \cup \{C_1, \dots, C_m\}$ then,*

$$\sum_{j=1}^m \kappa(C_j) = \sum_{Y \in L} \mu(X, Y) \kappa(Y)$$

The above statement is referred to as the *fundamental theorem of dissection theory* in [29, Theorem 1.2]. In a nutshell, the number of connected components of the complement (combinatorial Euler characteristic of the complement, to be precise) depend on a condition on the intersections and that condition turns out to be just the Euler relation. Moreover the combinatorial Euler characteristic is a valuation if and only if each face of the dissection is a finite, disjoint union of open topological cells (see [29, Lemma 1.1]). Some authors have also considered more general types of arrangements. For example, see [21, 22] where arrangements of topological spheres having homologically trivial chambers are studied.

The approach we took to prove Theorem 2.5 can be traced back to a paper of Blass and Sagan [3, Theorem 2.1] where they show how to evaluate the characteristic polynomial of subarrangements of the type B braid arrangement. Ehrenborg and Readdy [10] generalized this work and used it to determine the characteristic polynomial of any subspace arrangement defined over an infinite field. They explicitly used the Gromer's integral theorem to prove that the characteristic polynomial is a valuation. While studying toric arrangements with M. Slone [9] they generalized the previous result and proved the above mentioned Corollary 2.7. The idea of looking at the Euler characteristic as an integral of indicator functions is due to Chen [4], see also [5].

Finally we compare our strategy with that of Zaslavsky's, used to prove the fundamental theorem of dissection theory in [29]. In order to do this we will briefly sketch the outline of his proof. At the very foundation of both strategies lies the idea of using the combinatorial Euler characteristic as a valuation. In order to implement this idea we have used Möbius inversion whereas Zaslavsky has used a technical property of valuations. But before that a few definitions.

Definition 2.9. Let \mathcal{D} be a distributive lattice and R be a unitary ring. Let $M(\mathcal{D}, R)$ denote the free R -algebra whose basis is the elements of \mathcal{D} and the multiplication on the basis elements is defined by setting $xy = x \wedge y$ and then extended by linearity. In this algebra the set $N(\mathcal{D}, R)$ of all linear combinations of elements of the form $x \vee y + x \wedge y - x - y$ is an ideal. The *valuation ring* of \mathcal{D} over R is defined to be the quotient $M(\mathcal{D}, R)/N(\mathcal{D}, R)$ and denoted by $V(\mathcal{D}, R)$.

Definition 2.10. Let \mathcal{P} be a finite poset and R be a unitary ring. The *Möbius algebra* $M(\mathcal{P}, R)$ of \mathcal{P} is the free R -module whose basis is the elements of \mathcal{P} , with a product defined by

$$xy := \sum_{t \mid t \leq x, y} e_t(\mathcal{P}), \quad \forall x, y \in \mathcal{P}$$

and extended to $M(\mathcal{P}, R)$ by linearity, where we define

$$e_t(\mathcal{P}) := \sum_{s \in \mathcal{P}} \mu_{\mathcal{P}}(s, t) s, \quad \forall t \in \mathcal{P}.$$

Note that the elements of the type $e_t(\mathcal{P})$ are orthogonal idempotents in $M(\mathcal{P}, R)$ and they also form a basis of this algebra. Moreover for a distributive lattice the two definitions of

$M(\mathcal{D}, R)$ agree. Zaslavsky proves a theorem that establishes a relationship between the valuation ring of D and the canonical idempotents $e_t(\mathcal{P})$ of a subset of \mathcal{D} . Instead of stating this theorem we will state an important and relevant consequence.

Theorem 2.11. *Let ϕ be a valuation of the finite distributive lattice \mathcal{D} , and let \mathcal{P} be a subset of \mathcal{D} containing $\hat{0}$ and every join-irreducible element. Then, for any $t \in \mathcal{P}$ which is not $\hat{0}$ or a join-irreducible element of \mathcal{D} ,*

$$\sum_{s \in \mathcal{P}; s \leq t} \mu_{\mathcal{P}}(s, t) \phi(s) = 0.$$

In order to apply this theorem to dissection theory note that \mathcal{D} is the lattice of sets generated by the intersection poset and all the chambers. Theorem 2.8 then follows once we use the valuation κ .

3. FACES OF AN ARRANGEMENT

In this section we use formula (2.4) to find the number of various dimensional faces of an arrangement. We start with a definition.

Definition 3.1. The f -vector of an arrangement of submanifolds in a l -dimensional manifold is the vector

$$f = (f_0, f_1, \dots, f_l) \in \mathbb{N}^{l+1}$$

where f_k denotes the number of k -dimensional faces of the arrangement.

Theorem 2.5 can be used to count the number of faces (of all dimensions) of an arrangement. Note that these faces are the chambers of the restricted arrangements defined as follows. For $Y \in L(\mathcal{A})$, the arrangement restricted to Y is

$$\mathcal{A}^Y := \{N \cap Y \mid N \in \mathcal{A} \text{ and } \emptyset \neq N \cap Y \neq Y\}$$

Theorem 3.2. *Let X be a smooth, real manifold of dimension l and \mathcal{A} be an arrangement of submanifolds. Then the numbers f_k , are given by*

$$(3.1) \quad f_k = \sum_{\dim Y=k} (-1)^k \left(\sum_{\substack{Z \in L \\ Y \leq Z}} \mu(Y, Z) \kappa(Z) \right)$$

Proof. As $\mathcal{F}(\mathcal{A}) = \{\mathcal{C}(\mathcal{A}^Y) \mid Y \in L\}$ the number of k -faces of \mathcal{A} is given by

$$f_k = \sum_{\substack{Y \in L(\mathcal{A}) \\ \dim Y=k}} |\mathcal{C}(\mathcal{A}^Y)|$$

Use (2.4) to substitute for $|\mathcal{C}(\mathcal{A}^Y)|$ in the above formula and note that

$$L(\mathcal{A}^Y) = \{Z \in L(\mathcal{A}) \mid Y \leq Z\}$$

By convention, these f_k 's are also expressed as the coefficients of the following *generating polynomial*

$$\sum_{k=0}^l f_k x^{l-k} = (-1)^l \sum_{Z \in L(\mathcal{A})} \kappa(Z) \sum_{Y \leq Z} \mu(Y, Z) (-x)^{l-\dim(Y)}$$

□

Recall that a finite lattice L is called *atomic* if every $x \in L$ is a join of atoms (the elements that cover $\hat{0}$). A lattice is called *semimodular* if x, y both cover $x \wedge y$ then $x \vee y$ covers both x and y . A *geometric* lattice is both atomic and semimodular (see for example [27, Chapter 3]). It is a well known fact that the intersection lattice of central hyperplane arrangement is geometric. The intersection poset of a submanifold arrangement need not be a lattice but every interval in it is a geometric lattice as proved in the following lemma.

Lemma 3.3. *Let \mathcal{A} be an arrangement of submanifolds in a l -manifold X . Then every interval of the intersection poset $L(\mathcal{A})$ is a geometric lattice.*

Proof. Consider an interval $[Y, Z]$ in $L(\mathcal{A})$, such that $\dim Y = i$ and $\dim Z = j$. There exists an open set V in X and a coordinate chart ϕ such that $\phi(V \cap Y)$ is homeomorphic to an i -dimensional subspace of \mathbb{R}^l . Moreover, $\{\phi(N \cap V) | N \in \mathcal{A}^Y\}$ is a central arrangement of hyperplanes in \mathbb{R}^i . For any $W \in [Y, Z]$ the subspace $\phi(W \cap V)$ is homeomorphic to a subspace of \mathbb{R}^i that contains $\phi(Z \cap V)$. In particular, the $(i-1)$ -dimensional subspaces in $[Y, Z]$ map to hyperplanes in \mathbb{R}^i that contain $\phi(Z \cap V)$. This correspondence gives us an essential central arrangement of hyperplanes in \mathbb{R}^{i-j} when we quotient out $\phi(Z \cap V)$. This correspondence is also a poset isomorphism and hence $[Y, Z]$ is a geometric lattice. Moreover for geometric lattices the Möbius function alternates in sign and consequently $(-1)^{\dim Y - \dim Z} \mu(Y, Z) > 0$ (see [27, Proposition 3.10.1]). \square

We say that the submanifolds X are in *general position* if intersection of any i of the submanifolds, $i \geq 1$, is either empty or $(l-i)$ -dimensional. In the case of general position arrangements, every interval of $L(\mathcal{A})$ is a Boolean algebra hence the rank generating function for an interval $[Y, Z]$ is $(x+1)^{l-\dim Z}$ and also $\mu(Y, Z) = (-1)^{\dim Y - \dim Z}$ (see [27, Example 3.8.3]).

Now we compute f -vectors for submanifold arrangement in some particular manifolds. In each of the following cases we substitute the appropriate values of κ in the equation 3.1 and also use the above stated facts regarding the Möbius function. For similar calculations also see [9, 29]. In the following examples, as usual \mathcal{A} denotes the arrangement of submanifolds, $L(\mathcal{A})$ is its intersection poset. Finally, let a_k denote the number of elements of rank k in intersection poset.

Example 3.4. If $X \cong \mathbb{R}^l$ and \mathcal{A} is an arrangement of hyperplanes then

$$f_k = \sum_{\dim Y=k} \sum_{Y \leq Z} |\mu(Y, Z)|.$$

If the hyperplanes are in general position then

$$f_k = \sum_{j=0}^k a_j \binom{l-j}{l-k}.$$

Example 3.5. Now consider an arrangement of codimension 1 subtori in an l -torus,

$$f_k = \sum_{\substack{\dim Y=k \\ \dim Z=0 \\ Y \leq Z}} |\mu(Y, Z)|.$$

If the subtori are in general position then the number f_k takes the following simpler form

$$f_k = a_0 \binom{l}{l-k}.$$

Recently there have been a lot of interest in toric arrangements. See [9, 16, 19] for similar formulas and other types of enumeration problems in a torus.

Example 3.6. If $X \cong S^l$ an l -dimensional sphere and \mathcal{A} is an arrangement of hyperspheres (codimension 1 sub-spheres or equatorial spheres) such that all the intersections are lower dimensional spheres then

$$f_k = 2 \sum_{\dim Y=k} \sum_{\substack{Y \leq Z \\ \dim Z \geq 2 \\ \text{even}}} |\mu(Y, Z)| + \sum_{\dim Y=k} \sum_{\substack{Y \leq Z \\ \dim Z=0}} |\mu(Y, Z)|.$$

If all of the hyperspheres are in general position then

$$f_k = 2 \sum_{j=2, \text{even}}^k a_j \binom{l-j}{l-k} + a_0 \binom{l}{k}.$$

Example 3.7. Let X be the l -dimensional projective space and \mathcal{A} be an arrangement of $l-1$ -dimensional projective space

$$f_k = \sum_{\dim Y=k} \sum_{\substack{Y \leq Z \\ \dim Z \geq 0 \\ \dim Z \text{ even}}} |\mu(Y, Z)|.$$

If all the $l-1$ projective spaces are in general position then

$$f_k = \sum_{j=0, \text{even}}^k a_j \binom{l-j}{l-k}.$$

We end this paper by describing a relationship between the intersection poset and the face poset, which extends a result due to Bayer and Sturmfels [1] for hyperplane arrangements. Let \mathcal{F}^* be the dual of the face poset, define the map $\psi: \mathcal{F}^* \rightarrow L(\mathcal{A})$ by sending each face to the smallest dimensional subspace in the intersection poset that contains the face. For oriented matroids this is the support map going to its underlying matroid. The map ψ is order and rank preserving, as well as surjective hence we will look at it as a map from the set of chains of \mathcal{F}^* to the set of chains of $L(\mathcal{A})$.

Theorem 3.8. *Let $c = \{Y_1 \leq Y_2 \leq \dots \leq Y_k\}$, $k \geq 2$, be a chain in the intersection poset $L(\mathcal{A})$ of an arrangement of submanifolds \mathcal{A} . Then the cardinality of the inverse image of the chain c under the map ψ is given by the following formula*

$$|\psi^{-1}(c)| = \prod_{i=1}^{k-1} \left(\sum_{Y_i \leq Z \leq Y_{i+1}} (-1)^{l-\dim Z} \mu(Y_i, Z) \right) \cdot |\mathcal{C}(\mathcal{A}^{Y_k})|$$

Proof. The arguments are similar to the proof of [9, Theorem 3.13]. The number of ways of selecting a face F_k such that $\psi(F_k) = Y_k$ is equal to the number of chambers of \mathcal{A}^{Y_k} . A face F_{k-1} is in $\psi^{-1}(Y_{k-1})$ if it is a chamber in the arrangement $\mathcal{A}^{Y_{k-1}}$ and also contains the face F_k . The number of such faces is equal to the number of chambers in the central hyperplane

arrangement whose intersection lattice is isomorphic to $[Y_{k-1}, Y_k]$. By repeating this process for all the subspaces up to Y_1 we get the desired formula. \square

CONCLUDING REMARKS

Usually the cell enumeration problems are studied in the context of cell complexes homeomorphic to either a disc or a sphere. For example, there is an abundant literature concerning f -vectors of polytopes, Dehn-Sommerville equations, the g -theorem for simplicial polytopes etc., see [11]. In this paper we consider cell complexes which are not necessarily homeomorphic to spheres. Studying the analogues of traditional enumeration problems (for polytopes) in this general context seems very interesting. In this section we outline some such questions. We direct the interested reader to [9, Section 5] for more open questions regarding regular subdivisions of manifolds.

It is well known that the polar dual of a central hyperplane arrangement is a zonotope. In fact, there is a one-to-one correspondence between zonotopes of dimension l , central hyperplane arrangements in \mathbb{R}^l and oriented matroids on rank l (see [2] for details). For submanifold arrangements the zonotopes are replaced by metrical-hemisphere (MH for short) complexes. MH-complex is a regular cell complex whose face poset behaves very much like that of a zonotope. For example, there is a product structure on the cells, the 2-cells are combinatorially equivalent to regular $2k$ -gons, MH-complex is dual to the cell structure induced by a submanifold arrangement etc. (see [7, Chapter 3]). It will be interesting to study enumeration problems for these complexes.

As stated earlier, Zaslavsky's theorem generalizes to oriented matroid and the formula counts the number of topes. Our current work in progress concerns finding a generalization of oriented matroids. The combinatorial structure arising from the face poset of an MH-complex resembles that of an oriented matroid. Also, it is possible to find an analogue of matroids in the setting of submanifold arrangements (very similar to multiplicity matroids discovered in [19]). The aim here is to come up with a Tutte polynomial which explains the deletion-restriction phenomenon in general this setting and also incorporates the characteristic polynomial we have defined.

Finally, one would also like to understand the combinatorial type of the chambers. It is not hard to see that closure of a chamber is a nice manifold with corners (i.e. every point has a neighborhood homeomorphic to a neighborhood of the origin in $[0, \infty)^r \times \mathbb{R}^l$). It can be shown that in some cases the face poset of a closed chamber is isomorphic to that of a convex polytope. Given an arrangement of submanifolds is it possible to characterize the bounded chambers that are combinatorially simplices or cubes or any other nice polytopes?

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